9.4, 9.5 WS/Quiz solutions

1 Problem 1

1. (a) Show that $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$.

(b) Write out the 5th partial sum of $\sum_{i=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$ then find its numerical value.

(c) Find the sums of $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$ and $\sum_{n=12}^{\infty} \frac{2}{(n+1)(n+3)}$.

(a) Show that $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$.

We solve the partial fraction decomposition

$$\frac{2}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

so $2 = A(n+3) + B(n+1)$. Then letting $n = -1, -3$, we find that $A = 1$, $B = -1$ respectively.

so

$$\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$$

(b) Write out the 5th partial sum of $\sum_{i=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$ then find its numerical value.

Note

$$\begin{aligned}
S_1 &= \frac{1}{1+1} - \frac{1}{1+3} = \frac{1}{2} - \frac{1}{4} && (1)\\
S_2 &= a_1 + a_2 = S_1 + a_2 = S_1 + \frac{1}{2+1} - \frac{1}{2+3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} && (2)\\
S_3 &= S_2 + a_3 = S_2 + \frac{1}{3+1} - \frac{1}{3+3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + (\frac{1}{4} - \frac{1}{6}) = \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6} && (3)\\
S_4 &= S_3 + a_4 = S_3 + \frac{1}{5} - \frac{1}{7} = \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6} + (\frac{1}{5} - \frac{1}{7}) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} && (4)\\
S_5 &= S_4 + a_5 = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} + (\frac{1}{6} - \frac{1}{8}) = \frac{1}{2} + \frac{1}{3} - \frac{1}{7} - \frac{1}{8} = \frac{3}{8} + \frac{4}{21} = \frac{63-32}{164} = \frac{31}{164} && (5)\\
& && (6)
\end{aligned}$$

(c) Find the sums of $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$ and $\sum_{n=12}^{\infty} \frac{2}{(n+1)(n+3)}$.

Note $S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$ for $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$.

Thus as $n \to \infty$, $S_n \to 1/2 + 1/3 = 5/6$.

Starting at 12 shifts us, we'll find the first few partial sums

$$S_1' = a_{12} = \frac{1}{12+1} - \frac{1}{12+3} = \frac{1}{13} - \frac{1}{15} \tag{7}$$

$$S_2' = a_{12} + a_{13} = \frac{1}{13} - \frac{1}{15} + (\frac{1}{14} - \frac{1}{16}) \tag{8}$$

$$S_3' = S_2 + a_{14} = \frac{1}{13} - \frac{1}{15} + \frac{1}{14} - \frac{1}{16} + (\frac{1}{15} - \frac{1}{17}) = \frac{1}{13} + \frac{1}{14} - \frac{1}{16} - \frac{1}{17} \tag{9}$$

$$\tag{10}$$

and the pattern will continue for $S_n' = \frac{1}{13} + \frac{1}{14} - \frac{1}{n+2} - \frac{1}{n+3}$ so that

$$\sum_{n=12}^{\infty} \frac{2}{(n+1)(n+3)} = \lim_{n\to\infty} S_n' = \frac{1}{13} + \frac{1}{14} = \frac{14+13}{182} = \frac{27}{182}$$

2 Problem 2

2. (a) Write 0.6363... as a geometric series and find its numerical value (hint: 0.636363...= 0.63+0.0063+0.000063+...)

(b) Write 0.999... as a geometric series and find its numerical value.

(c) Does (b) indicate 0.999...=1?

(a) Write 0.6363... as a geometric series and find its numerical value (hint: 0.636363...= 0.63+0.0063+0.000063+...)

Note $0.63 = 63 \cdot 1/100 = 63 \cdot 1/10^2$, $0.0063 = 63 \cdot 1/10000 = 63 \cdot 1/10^4$, and $0.000063 = 63 \cdot 1/1000000 = 63 \cdot 1/10^6$. In general,

$$0.636363... = 63 \cdot 1/10^2 + 63 \cdot 1/10^4 + 63 \cdot 1/10^6 + ... + 63 \cdot 1/10^{2n} + ... \tag{11}$$

$$= \sum_{i=1}^{\infty} 63 \cdot 1/10^{2n} = \sum_{i=1}^{\infty} 63 \cdot 1/(10^2)^n = \sum_{i=1}^{\infty} 63 \cdot 1/100^n = \sum_{i=1}^{\infty} 63 \cdot (1/100)^n \tag{12}$$

This is a geometric series with ratio $r = 1/100 < 1$, so the series converges to $\frac{\text{first term}}{1-\text{ratio}} = \frac{0.63}{1-1/100} = \frac{0.63}{99/100} = 0.63 \cdot 100/99 = 63/99 = 7/11$.

Thus

$$0.636363... = \sum_{i=1}^{\infty} 63 \cdot (1/100)^n = 7/11.$$

(b) Write 0.999... as a geometric series and find its numerical value.

Again, write

$$0.999... = 0.99 + 0.0099 + 0.000099 + ...$$

with $0.999... = \sum_{i=1}^{\infty} 99 \cdot (1/100)^i = \frac{\text{first term}}{1-\text{ratio}} = \frac{0.99}{1-1/100} = \frac{0.99}{99/100} = 1$.

(c) Does (b) indicate 0.999...=1?

Yes, our solution for (b) shows $0.999... = 1$.

3 Problem 3

3. When a superball is dropped onto a hardwood floor, it bounces up to approximately 80% of its original height. Suppose a superball is dropped from 5 feet above a hardwood floor.

(a) Find the distance (in feet) that the ball travels during three bounces (that is, until it reaches its maximum height after bouncing on the floor three times).

(b) Find a formula involving a geometric series for the distance the ball travels (in theory) if it is left to bounce forever. Then evaluate the sum of the series you found.

(a) Find the distance (in feet) that the ball travels during three bounces (that is, until it reaches its maximum height after bouncing on the floor three times).

The ball first travels 5m to the ground, then to the max height of the first bounce, b_1, will be 80% of 5m, i.e. $b_1 = 0.8 \cdot 5 = 4$ meters. Thus, the ball has traveled $5 + 4$ meters thus far. Then, the ball falls 4 meters, travelling $5 + 4 + 4 = 9$ meters thus far.

The max height of the second bounce, b_2, will be 80% of the max height of the first bounce. Thus $b_2 = 0.8 \cdot b_1 = 0.8 \cdot 4 = 3.2$ meters.

Thus the ball would travel a total distance of $5 + 4 + 4 + 3.2 + 3.2 = 15.4$ meters at the moment of the third bounce.

Since $b_3 = 0.8 \cdot b_2 = 0.8 \cdot 3.2 = 2.4$ meters is the max height of the third bounce, the total distance the ball travels to the maximum height after bouncing on the floor three times is $5 + 4 + 4 + 3.2 + 3.2 + 2.4$, or $5 + 2 \cdot b_1 + 2 \cdot b_2 + b_3$.

(b) In general, the total distance the ball travels until it reaches its max height of the n'th bounce is $b_0 + 2b_1 + 2b_2 + ... + 2b_{n-1} + b_n$. Also, note

$$b_n = 0.8b_{n-1} = 0.8(0.8b_{n-2}) = 0.8^2 b_{n-2} = ... = 0.8^i b_{n-i} = ... = 0.8^n b_0 = 0.8^n \cdot 5.$$

Then a formula for the total distance the ball bounces after being allowed to bounce forever is

$$\begin{aligned} b_0 + 2b_1 + 2b_2 + 2b_3 + ... = b_0 + 2(b_1 + b_2 + b_3 + ...) &= b_0 + 2(0.8b_0 + 0.8^2 b_0 + 0.8^3 b_0 + ...) && (13) \\ &= 5 + 2(0.8 \cdot 5 + 0.8^2 \cdot 5 + 0.8^3 \cdot 5 + ...) && (14) \\ &= 5 + 2 \cdot \sum_{i=1}^{\infty} (0.8^i \cdot 5) && (15) \end{aligned}$$

Note $\sum_{i=1}^{\infty} 0.8^i \cdot 5$ is a geometric series with ratio $r = 0.8 < 1$, so the series converges to

$$\sum_{i=1}^{\infty} (0.8^i \cdot 5) = \frac{\text{first term}}{1 - \text{ratio}} = \frac{0.8 \cdot 5}{1 - 0.8}.$$

This simplifies to $\frac{4}{0.2} = 20$. Then the total distance the ball bounces is $5 + 2(20) = 45$ meters.

4 Problem 5

From Exercise 48(a) on p. 602, we know that the numerical value of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

(a) Using this information, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} 1/n^{1.9}$ converges. If $\sum_{n=1}^{\infty} 1/n^{1.9}$ converges, what are the relative sizes of $\sum_{n=1}^{\infty} n^{1.9}$ $\sum_{n=1}^{\infty} 1/n^2$?

(b) Similarly, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} 1/n^{2.1}$ converges. If $\sum_{n=1}^{\infty} = 1/n^{2.1}$ converges, what are the relative sizes of$\sum_{n=1}^{\infty} 1/n^{2.1}$ and $\sum_{n=1}^{\infty} 1/n^2$?

(a) Using this information, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} 1/n^{1.9}$ converges. If $\sum_{n=1}^{\infty} 1/n^{1.9}$ converges, what are the relative sizes of $\sum_{n=1}^{\infty} n^{1.9}$ and $\sum_{n=1}^{\infty} 1/n^2$?

We cannot conclude that $\sum_{n=1}^{\infty} 1/n^{1.9}$ converges since $n^{1.9} < n^2$, so $\frac{1}{n^2} < \frac{1}{n^{1.9}}$. Thus

$$\sum_{n=1}^{\infty} 1/n^2 < \sum_{n=1}^{\infty} 1/n^{1.9}$$

so we cannot use the comparison test (we should compare our sum $\frac{1}{n^{1.9}}$ to a *larger* convergent series). We also cannot use the limit comparison test since $\lim_{n\to\infty} \frac{1/n^{1.9}}{1/n^2} = \lim_{n\to\infty} n^{0.1} = \infty$.

(b) Similarly, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} 1/n^{2.1}$ converges. If $\sum_{n=1}^{\infty} = 1/n^{2.1}$ converges, what are the relative sizes of$\sum_{n=1}^{\infty} 1/n^{2.1}$ and $\sum_{n=1}^{\infty} 1/n^2$?

We can conclude the convergence of $\sum_{n=1}^{\infty} 1/n^{2.1}$ based on $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ since we have $n^2 < n^{2.1}$, so $\frac{1}{n^{2.1}} < \frac{1}{n^2}$, and thus $\sum_{n=1}^{\infty} 1/n^{2.1} < \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and so

$$\sum_{n=1}^{\infty} 1/n^{2.1}$$

is convergent by the comparison test (a limit comparison test fail to apply here however)

5 Quiz 9.4, 9.5

Determine whether or not the series converges, and if so, find its sum:

$$1.\ \sum_{n=2}^{\infty}(0.33)^n \quad 2.\ \sum_{n=0}^{\infty} \frac{2^n+5^n}{2^n \cdot 5^n}$$

and use the comparison test or the integral test to determine whether the series converges or diverges:

$$3.\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad 4.\ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$$

1. Determine whether or not the series converges, and if so, find its sum: $\sum_{n=2}^{\infty}(0.33)^n$

This is a geometric series with ratio $r = 0.33 < 1$ and first term 0.33^2. Then

$$\sum_{n=2}^{\infty}(0.33)^n = \frac{\text{first term}}{1-\text{ratio}} = \frac{0.33^2}{1-0.33} = \frac{0.33^2}{0.67}.$$

2. Determine whether or not the series converges, and if so, find its sum: $\sum_{n=0}^{\infty} \frac{2^n+5^n}{2^n \cdot 5^n}$

Note

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n+5^n}{2^n \cdot 5^n} &= \sum_{n=0}^{\infty} \frac{2^n}{2^n \cdot 5^n} + \frac{5^n}{2^n \cdot 5^n} && (16)\\
&= \sum_{n=0}^{\infty} \frac{1}{5^n} + \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \frac{1}{2^n} && (17)\\
&= \frac{1}{1-1/5} + \frac{1}{1-1/2} = 5/4 + 2 = 13/4 && (18)\\
& && (19)
\end{aligned}
$$

By properties of convergent series, and the convergence of geometric series $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $|r| < 1$.

3. Use the comparison test or the integral test to determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

We can say it diverges by p-series since $p = 1/2$ does't satisfy $p > 1$. We could also compare it to the harmonic series $\sum_{n=1}^{\infty} 1/n$.

Since $\sqrt{n} < n$, so $\frac{1}{n} < \frac{1}{\sqrt{n}}$, so $\sum_{n\geq 1} 1/n < \sum_{n\geq 1} \frac{1}{\sqrt{n}}$, and since the harmonic series diverges, our sum $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges too.

4. Use the comparison test or the integral test to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+3}$$

Note that $\frac{\sqrt{n}}{n^2+3} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{2-1/2}} = \frac{1}{n^{3/2}}$ for $n \geq 1$.

Thus

$$\sum_{n\geq 1} \frac{\sqrt{n}}{n^2+3} < \sum_{n\geq 1} \frac{\sqrt{n}}{n^2} = \sum_{n\geq 1} \frac{1}{n^{3/2}}$$

and note $\sum_{n\geq 1} \frac{1}{n^{3/2}}$ converges by p-series, with $p = 3/2 > 1$. Thus by the comparison test, our series $\sum_{n\geq 1} \frac{\sqrt{n}}{n^2+3}$ is smaller than a convergent series, so it is convergent.